

Quantum State Preparation by Controlled Dissipation in Finite Time: From Classical to Quantum Controllers *

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Abstract

We propose a general scheme for dissipatively preparing arbitrary pure quantum states on a multipartite qubit register in a finite number of basic control blocks. Our “splitting-subspace” approach relies on control resources that are available in a number of scalable quantum technologies (complete unitary control on the target system, an ancillary resettable qubit and controlled-not gates between the target and the ancilla), and can be seen as a “quantum-controller” implementation of a sequence of classical feedback loops. We show how a large degree of flexibility exists in engineering the required conditional operations, and make explicit contact with a stabilization protocol used for dissipative quantum state preparation and entanglement generation in recent experiments with trapped ions.

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1 Introduction and Notation

State preparation problems are of vital importance across quantum information processing applications, ranging from initialization of quantum computation and simulation algorithms in network-based as well as cluster-state architectures to use within enhanced quantum metrology protocols [12]. If the preparation is to be achieved irrespective of the initial state of the system, from a control standpoint this translates naturally into a stabilization problem. Physically, one is compelled to make the quantum evolution irreversible, by introducing open-system features in either an open-loop or closed-loop fashion [2]. Closed-loop control based on quantum measurements and feedback techniques provides, in particular, a very natural and powerful toolbox [20], with feedback from a single ancilla qubit together with fast, complete unitary control allowing in principle to engineer arbitrary open-system dynamics [9].

Quantum state stabilization problems have been studied in depth for continuous-time dynamical models, for either Markovian “output feedback” [18, 15, 16] or strategies based on state reconstruction by quantum filtering [17, 10]. It is worth remarking that in such continuous-time scenarios the target state can typically be reached only asymptotically, in the (formal) limit of infinite evolution time. Recently, a linear-algebraic framework for analysis and synthesis has also been developed for the discrete-time case. In particular, for a given indirect quantum measurement and complete unitary control over the target

system, it has been shown that pure states are generically stabilizable [4]. This is always true for a projective measurement, and if the latter is associated to a non-degenerate observable, it further follows that the desired state can be prepared in a *single* step of measurement-plus-control [5].

From an implementation point of view, a problem associated with measurement-based feedback schemes is that the required control resources need not be readily available for many state-of-the-art experimental quantum devices. In this work, we investigate how to prepare a desired pure state in *finite time* by means of a reasonable set of resources for a multipartite qubit system. In particular, while still assuming complete unitary control, we shall effectively “encode” the whole feedback loop in a finite sequence of *coherent feedback* actions [7, 19], where the controller is itself a quantum system (a qubit) and no measurement is involved. For certain experimental settings, most notably trapped ions, these control resources are not only achievable in principle, but have been already experimentally demonstrated up to 5 qubits [3]. In particular, we will illustrate how the “stabilizer pumping” strategy proposed in [3, 11] fits into our general “splitting-subspace” framework, and how control actions achieving stabilization of arbitrary pure target states can be explicitly synthesized.

We begin by recalling some basic concepts and notations. Let S be a *finite-dimensional* quantum system of interest, with associated Hilbert space $\mathcal{H} \sim \mathbb{C}^d$. Vectors and linear functionals on \mathcal{H} are denoted using Dirac’s notation with $|\psi\rangle$ and $\langle\phi|$, respectively [13]. Observable quantities on S are associated to self-adjoint operators on \mathcal{H} , here represented by Hermitian matrices $X = X^\dagger \in \mathfrak{H}(\mathcal{H})$. In particular, the *state* of S is in general described by a *density operator* $\rho \in \mathfrak{D}(\mathcal{H}) = \{\rho \in \mathfrak{H}(\mathcal{H}) | \rho \geq 0, \text{tr}(\rho) = 1\}$, with *pure states* corresponding to the extreme point of the (convex) set $\mathfrak{D}(\mathcal{H})$. Unitary matrices are denoted by $U \in \mathcal{U}(\mathcal{H})$. The (real) spectrum of an observable X represents the set of the possible outcomes in the simplest case of a so-called projective (or von Neumann’s) quantum measurement on S . Suppose that X admits a spectral decomposition of the form $X = \sum_i x_i \Pi_i$, in terms of a complete set of orthogo-

nal projectors $\{\Pi_i\}$ on \mathcal{H} . According to the basic postulates for von Neumann’s measurements, the probability of obtaining x_i given the pre-measurement state ρ is $p_i = \text{tr}(\Pi_i \rho) = \text{tr}(\Pi_i \rho \Pi_i)$. Conditionally upon the measurement outcome x_i being recorded, the (normalized) post-measurement state of S then becomes $\rho|_i = \frac{1}{p_i} \Pi_i \rho \Pi_i$.

If S consists of multiple (distinguishable) *subsystems* S_j , $j = 1, \dots, N$, each associated to a Hilbert space \mathcal{H}_j , the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \dots \otimes \mathcal{H}_N$ [12], and observables and density operators remain associated with Hermitian and positive-semidefinite, trace-one operators on \mathcal{H} , respectively. Given a state $\rho \in \mathfrak{D}(\mathcal{H})$ of S , the *reduced state* of one (or a subset) of subsystems may be uniquely determined by taking the partial trace over the remaining subsystem(s), for instance in the simplest bipartite setting, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, we shall denote by $\rho_1 = \text{tr}_{\mathcal{H}_2}(\rho)$ the reduced density operator describing the first subsystem alone.

If information about a quantum system is gathered indirectly, through measurements of a correlated auxiliary system, the formalism of von Neumann’s projective measurements is overly restrictive in general and a description of the effect of the measurement on the system of interest is provided by so-called *generalized measurements*. To a set of possible measurement outcomes labeled by k , we associate a set of measurement operators $\{M_k\}$ on the system of interest, in such a way that $\sum_k M_k^\dagger M_k = I$, with I being the identity operator. The probability of obtaining the k -th outcome is then computed as $p_k = \text{tr}(M_k^\dagger M_k \rho)$, ρ denoting now the reduced state of the system of interest which, after the outcome is recorded, is updated to $\rho|_k = \frac{1}{p_k} M_k \rho M_k^\dagger$.

By taking the average over the possible outcomes, we obtain a Trace-Preserving Completely Positive (TPCP) linear transformation of the state in the form of a so-called Kraus map [6], that is,

$$\rho \mapsto \mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger.$$

The standard case of projective measurements on S is formally recovered by choosing $M_k = \Pi_k$.

2 Discrete-Time Feedback Stabilization

Suppose that a generalized measurement operation can be performed on the target system at times $t = 1, 2, \dots$, resulting in an open-system, discrete-time dynamics described by a Kraus map with operators $\{M_k\}$. Suppose, in addition, that we are able to enact arbitrary unitary control actions on the state of S , that is, $\rho \mapsto U\rho U^\dagger$, for arbitrary $U \in \mathcal{U}(\mathcal{H})$, with unitary manipulations that are fast with respect to the measurement time scale, so that the measurement and the control effectively act in distinct “time slots”.

We can then in principle implement a Markovian feedback law, consisting in a map from the set of measurement outcomes to the set of unitary matrices,

$$U(k) : k \mapsto U_k \in \mathcal{U}(\mathcal{H}).$$

If the above measurement-control loop is iterated and we average over the measurement results at each step, the net result is a different TPCP map, which describes the evolution of the state *immediately after* each application of the controls:

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger.$$

Controllability and stabilizability for the resulting class of discrete-time, closed-loop dynamics have been studied in detail in [9, 5, 4, 1]. In particular, from the results of [4, 1], it is immediate to see that if the following control resources are available:

- (f1) *Arbitrary unitary control actions* $\{U_k\}$;
- (f2) *A non-degenerate projective measurement*, associated to a resolution of the identity on S , $\{\Pi_k = |\phi_k\rangle\langle\phi_k|\}_{k=1}^d$,

then the system can be prepared in *any desired pure state in one step*. Let $\rho_d = |\psi\rangle\langle\psi|$ denote the target pure state: the desired preparation is then simply achieved by choosing control operations U_k such that $U_k|\phi_k\rangle = |\psi\rangle$. In control-theoretic terms, this effectively implements a quantum *dead-beat* controller, reaching the desired state not just asymptotically but

in one step. The above is indeed an abstract description of the most straightforward procedure to prepare a given state: first, measure the system projecting it onto some known state, and then enact some open-loop, controlled transition to steer it to the desired state. Despite its conceptual simplicity, this strategy may become challenging in practical control scenarios where the required measurement procedures are unavailable (for instance, measurement may be destructive, too slow and/or inaccurate, or not having the needed resolution). In what follows, by focusing on multi-qubit systems, we show how to achieve the same pure-state preparation by replacing the above full resolution measurement with the ability of using an extra qubit as a fully coherent resettable quantum controller [7, 8].

3 The Splitting-Subspace Approach

Consider an N -qubit register, with associated Hilbert space $\mathcal{H}_Q = \bigotimes_j \mathcal{H}_j \sim \mathbb{C}^{2^N}$, and with $\{|\phi_j\rangle\}_{j=1}^{2^N}$ denoting the standard (computational) basis of \mathcal{H}_Q , $|\phi_1\rangle = |0\dots 00\rangle$, $|\phi_2\rangle = |0\dots 01\rangle$, $|\phi_3\rangle = |0\dots 10\rangle$, and so on.

Assume that the following control resources are available:

- (s1) *Arbitrary unitary control actions* $\{U\}$ on the N target qubits;
- (s2) *An auxiliary control qubit*, with Hilbert space \mathcal{H}_c , that can be reset to a known pure state, say $|1\rangle$;
- (s3) *Controlled-not unitaries* $C_{\text{in}}, C_{\text{out}} \in \mathcal{U}(\mathcal{H}_c \otimes \mathcal{H}_j)$ between the control and one of the target qubit. Without loss of generality, we can pick the first qubit ($j = 1$) and write $C_{\text{out}} = I_2 \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1|$, $C_{\text{in}} = |1\rangle\langle 1| \otimes I_2 + |0\rangle\langle 0| \otimes \sigma_x$, respectively.

We next show that a simple approach can be developed in order to design quantum circuits able to prepare the target pure state $\rho_d = |\psi\rangle\langle\psi|$ in a finite number of iterations. The first step is to provide a

characterization of the target state in terms of a family of *splitting subspaces*.

Lemma 3.1 (Splitting Subspaces) *Any N -qubit pure state $|\psi\rangle \in \mathcal{H}_Q$ can be described as the unique unit-norm vector in the intersection of N subspaces \mathcal{S}_k of dimension 2^{N-1} , that is,*

$$\text{span}\{|\psi\rangle\} = \bigcap_{k=1}^N \mathcal{S}_k. \quad (1)$$

Proof. It suffices to provide an explicit way to construct the \mathcal{S}_k , see also Figure 1 for illustration. Start by relabeling $|\psi_1\rangle := |\psi\rangle$ and complete it with $2^N - 1$ vectors so that $\{|\psi_k\rangle\}_{k=1}^{2^N}$ is an orthonormal basis¹. Next, define $\mathcal{S}_1 = \text{span}\{|\psi_k\rangle, k = 1, \dots, 2^{N-1}\}$, $\mathcal{S}_2 = \text{span}\{|\psi_k\rangle, k = 1, \dots, 2^{N-2}, 2^{N-1} + 1, \dots, 2^{N-1} + 2^{N-2}\}$, and so on for the remaining subspaces. Formally, by defining the matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Pi^{(k)} = I_2^{\otimes(k-1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_2^{\otimes(n-k)},$$

in the $|\psi_k\rangle$ -basis, we can identify the k -th splitting subspace as

$$\mathcal{S}_k = \text{range}(\Pi^{(k)}).$$

By construction, the only vector in the intersection of all such subspaces is $|\psi_1\rangle$. \square

For the important class of *stabilizer states* [12], a natural choice of splitting subspaces is provided by the one-dimensional joint eigenspaces of the associated stabilizer operators. However, as showed in the above Lemma, there is no need to restrict to this subset of pure states.

We also remark that in control-theoretic terms, requirement (s1) corresponds to complete controllability of the N -qubit in finite time at the level of its Hamiltonian description [2]. In the context of quantum computation, this is equivalent to requiring access to a (continuous) set of gates achieving exact

¹Clearly, there is a lot of freedom in doing so, effectively equivalent to choosing an element of $SU(2^N - 1)$. This freedom can be exploited in reducing the complexity of the unitary operations in what follows.

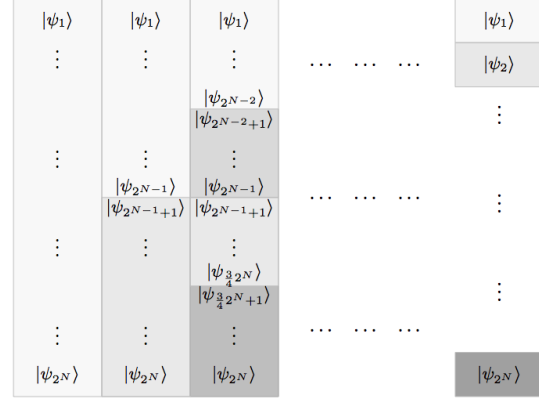


Figure 1: Pictorial representation of the constructive argument used in Lemma 3.1 to obtain a splitting-subspace description for the target state $|\psi_1\rangle = |\psi\rangle$.

universality [12]. In practice, if unitary operations are built out of a *discrete* set of universal gates, an arbitrary target gate U may be implemented by a finite-length quantum circuit only within a non-zero accuracy. In this case, the finite accuracy of the unitary control action will carry over to the stabilized state, with a worst-case error growing linearly in the number of steps.

3.1 Stabilization protocols

Building on the above result, the next step is to show that the state of the target qubits can be prepared in any splitting subspace \mathcal{S} in one iteration given the control resources specified in (s1)-(s3). If we could perform a projective measurement of $\Pi_{\mathcal{S}}, \Pi_{\mathcal{S}}^{\perp}$, the orthogonal projection onto \mathcal{S} and its orthogonal complement respectively, it would suffice to use a feedback law that does nothing if $\Pi_{\mathcal{S}}$ is measured, whereas for the other outcome it applies a U_{\perp} obeying

$$U_{\perp} \Pi_{\mathcal{S}}^{\perp} U_{\perp}^{\dagger} = \Pi_{\mathcal{S}}. \quad (2)$$

Notice, again, that such a control action would be highly not unique, since there is at least enough freedom as associated to the unitary mapping from \mathcal{S}^{\perp} to \mathcal{S} , which is an element of $SU(2^{N-1})$. The average

total evolution would then be:

$$\rho(t+1) \equiv \mathcal{E}_S[\rho(t)] = \Pi_S \rho(t) \Pi_S + U_\perp \Pi_S^\perp \rho(t) \Pi_S^\perp U_\perp^\dagger, \quad (3)$$

and it is easy to see that the support of $\rho(t+1)$ is contained in \mathcal{S} , as desired:

$$\begin{aligned} \text{tr}[\Pi_S \rho(t+1)] &= \text{tr}[\Pi_S \rho(t)] + \text{tr}[U_\perp^\dagger \Pi_S U_\perp \Pi_S^\perp \rho(t) \Pi_S^\perp] \\ &= \text{tr}[\Pi_S \rho(t)] + \text{tr}[\Pi_S^\perp \rho(t)] = 1, \end{aligned}$$

where we have taken advantage of Eq. (2).

Fortunately, the TPCP map \mathcal{E}_S in Eq. (3) can also be implemented without requiring projective measurements, by using the extra qubit as a quantum controller. Let $\{|\psi_k\rangle\}$ denote an orthonormal basis for \mathcal{H}_Q , such that the first 2^{N-1} elements are a basis for \mathcal{S} . Then proceed as follows:

- (1) Perform a unitary U_ψ^\dagger on the target qubits, where $U_\psi|\phi_k\rangle = |\psi_k\rangle$. In this way, the state of (in particular) the first qubit contains information on whether or not the state of the system is in \mathcal{S} .
- (2) Perform C_{out} on the control and the first qubits, thereby mapping the information of the first qubit on the state of the control qubit;
- (3) Perform C_{in} on the control and the first qubits, thereby changing the state of the first qubit depending on the projection of the control qubit on $|0\rangle$;
- (4) Perform the change of basis U_ψ to return to the original basis and (if needed for successive steps) reset the control qubit back to $|1\rangle$.

The net effect of operations (1)–(4) is the following unitary transformation on $\mathcal{H}_c \otimes \mathcal{H}_Q$:

$$\begin{aligned} U_{\text{tot}} &= U_\psi C_{\text{in}} C_{\text{out}} U_\psi^\dagger \\ &= U_\psi C_{\text{in}} U_\psi^\dagger U_\psi C_{\text{out}} U_\psi^\dagger \\ &= (|1\rangle\langle 1| \otimes I_{2^N} + |0\rangle\langle 0| \otimes U_\perp)(I_2 \otimes \Pi_S + \sigma_x \otimes \Pi_S^\perp), \end{aligned}$$

where again U_\perp satisfies Eq. (2). Thus, by tracing over the control degrees of freedom and assuming that

above steps (1)–(4) can be realized in one time unit, this yields:

$$\begin{aligned} \rho(t+1) &= \text{tr}_{\mathcal{H}_c} \left\{ U_{\text{tot}} [|1\rangle\langle 1| \otimes \rho(t)] U_{\text{tot}}^\dagger \right\} \\ &= \Pi_S \rho(t) \Pi_S + U_\perp \Pi_S^\perp \rho(t) \Pi_S^\perp U_\perp^\dagger = \mathcal{E}_S(\rho_t), \end{aligned}$$

as claimed. The whole sequence of the 4 operations (1)–(4) above will be referred to in the following as a *control step*, associated to a coherent implementation of a single feedback loop in discrete-time. If we have N splitting subspaces constructed as in the proof of Lemma 3.1, the basis $\{|\psi_k\rangle\}$ to be used at the beginning of the above protocol can be chosen *at each control step* to be a reordering of the first one. With this choice, the desired state-preparation result can then be easily established.

Theorem 3.1 *Any N -qubit pure state $\rho_d = |\psi\rangle\langle\psi|$ can be deterministically prepared using control resources (s1)–(s3) in N steps, irrespective of the initial state ρ .*

Proof. Let $\{\mathcal{S}_\ell\}$ be a splitting-subspace description for $|\psi\rangle$ as in Eq. (1), with $\mathcal{E}_{\mathcal{S}_\ell}$ the associated TPCP map given in (3). Now implement the map $\mathcal{E}_{\mathcal{S}_N} \circ \dots \circ \mathcal{E}_{\mathcal{S}_1}(\rho)$. By construction, each $\mathcal{E}_{\mathcal{S}_\ell}$ maps $\left(\bigcap_{k=1}^{\ell-1} \mathcal{S}_k\right) \cap \mathcal{S}_\ell^\perp$ onto $\left(\bigcap_{k=1}^{\ell-1} \mathcal{S}_k\right) \cap \mathcal{S}_\ell$. $\left(\bigcap_{k=1}^N \mathcal{S}_k\right) = \text{span}\{|\psi\rangle\}$. Thus, at the N -th control step, any initial state is driven into $\left(\bigcap_{k=1}^N \mathcal{S}_k\right) = \text{span}\{|\psi\rangle\}$. \square

We stress that, while the controlled-not operations entangling the control and the first qubits, as well as the resetting of the control qubit, are the same at each step (thus in real-world implementations will each take a fixed time to be enacted), *the U_ψ will in general differ* at different steps, since U_ψ will include a reshuffling of the basis elements in order to obtain the “nested” subspace structure used for the proof. Since, in general, one can see that up to $(N - \ell)/2$ qubit swaps are needed at step ℓ , this can lengthen the total implementation time, making the initial choice of the (non-unique) basis $\{|\psi_k\rangle\}$ important for efficient implementation.

3.2 Effective entangling operations

In practice, engineering exactly the controlled-not gates C_{in} and C_{out} we have assumed may be hard to achieve. Luckily, as it turns out, the way of realizing the “feedback” map (3) with quantum controllers is highly not unique. Even if the controlled-not gates are achievable in principle, different choices may be more convenient to implement, motivating us to further characterize the degree of flexibility intrinsic to our protocol. More precisely, it is easy to verify by direct calculation that *any pair* of entangling gates of the following form can achieve the desired task up to some additional unitary operation on the target qubits:

$$\begin{aligned}\tilde{C}_{\text{in}} &= (I_2 \otimes \tilde{V}_1) (|1\rangle\langle 1| \otimes I_2 + |0\rangle\langle 0| \otimes U_\perp) W^\dagger (I_2 \otimes \tilde{V}_2), \\ \tilde{C}_{\text{out}} &= (I_2 \otimes \tilde{U}_1) W (D_c \otimes \Pi_0 + O_c \otimes \Pi_1^\perp) (I_2 \otimes \tilde{U}_2),\end{aligned}\quad (4)$$

where D_c , (O_c) are unitary and diagonal (off-diagonal) in the standard basis of \mathcal{H}_c , \tilde{U}_j and \tilde{V}_j are unitary operators on \mathcal{H}_Q and W is an arbitrary unitary operator on the whole system, respectively. Π_0 (Π_1) indicates the orthogonal projections on the system subspaces corresponding to the first qubit being 0 (1). By invoking assumption (s1), \tilde{U}_j and \tilde{V}_j can be compensated by suitable unitary control: if, in the steps (2) and (3) of the protocol described above, we now apply $(I_2 \otimes \tilde{U}_1^\dagger) \tilde{C}_{\text{out}} (I_2 \otimes \tilde{U}_2^\dagger)$ and $(I_2 \otimes \tilde{V}_1^\dagger) \tilde{C}_{\text{in}} (I_2 \otimes \tilde{V}_2^\dagger)$, respectively, it follows that

$$\begin{aligned}\tilde{U}_{\text{tot}} &= U_\psi (I_2 \otimes \tilde{V}_1^\dagger) \tilde{C}_{\text{in}} (I_2 \otimes \tilde{V}_2^\dagger) (I_2 \otimes \tilde{U}_1^\dagger) \tilde{C}_{\text{out}} (I_2 \otimes \tilde{U}_2^\dagger) U_\psi^\dagger \\ &= (|1\rangle\langle 1| \otimes I_{2^N} + |0\rangle\langle 0| \otimes U_\perp) (D_c \otimes \Pi_S + O_c \otimes \Pi_S^\perp).\end{aligned}$$

The net reduced dynamics on the system is not affected by the phases introduced on the ancilla by D_c , O_c , hence tracing out the ancilla we recover (3) again.

In general, failing to compensate the action of \tilde{U}_j, \tilde{V}_j as described could potentially slow down (or even prevent) the desired state preparation since it need no longer be true that $(\bigcap_{k=1}^{\ell-1} \mathcal{S}_k) \cap \mathcal{S}_\ell^\perp$ is mapped onto $(\bigcap_{k=1}^{\ell-1} \mathcal{S}_k) \cap \mathcal{S}_\ell$, a key step in the convergence proof of Theorem 3.1. Such compensation may, however, be unnecessary if additional conditions

are obeyed. For instance, if $\Pi_{\mathcal{S},k}$ is the projector on the k -th splitting subspace \mathcal{S}_k , and it holds that:

$$\tilde{V}_1 \Pi_{\mathcal{S},k} = 0, \quad (5)$$

$$\tilde{V}_1 \Pi_{\mathcal{S},k} = \Pi_{\mathcal{S},k}, \quad (6)$$

then there is no need to compensate for \tilde{V}_1 . In fact, (5) ensures \mathcal{S}_k is still an invariant stable subspace for the current control step, and (6) that \tilde{V} acts as the identity on it, without compromising the result of the previous steps. A concrete example will be provided in the following section. Similar conditions can be worked out for $\tilde{U}_1, \tilde{U}_2, \tilde{V}_2$.

4 Case Study: Trapped Ions

4.1 Experimental Bell-state pumping

Recently, a toolbox for engineering the open-system dynamics of up to five qubits has been experimentally demonstrated using ^{40}Ca ions trapped in a linear geometry [3]. The control resources specified in (s1)–(s3) are available through a combination of single- and multi-qubit gates ensuring universal scalable ion-trap quantum computation, plus access to a controllable dissipative mechanism on an extra ancilla qubit, realized via a combination of optical pumping and spontaneous emission. Here, we revisit the implemented two-qubit “Bell-state pumping” protocol in the light of our splitting-subspace approach. Experimentally, a similar approach has been successfully employed to create GHZ states on up to 4 qubits. Let us denote the four Bell states as

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (7)$$

The system, initially in an unknown state specified by a density operator ρ , is deterministically prepared in the Bell state $|\Psi^-\rangle$ by realizing in two steps a quantum operation $\rho \mapsto |\Psi^-\rangle\langle\Psi^-|$. It is well known that Bell states are simple examples of stabilizer states²: each of the four Bell states in (7) may be uniquely

²Following standard notation [12], we shall denote by X_k, Y_k, Z_k the multi-qubit operators that act as Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ on the k -th qubit, and as identity on the rest.

characterized as a joint eigenstate (with eigenvalues ± 1) of a mutually commuting set of two stabilizer generators, for instance $Y_1 Y_2$ and $X_1 X_2$. The considered strategy engineers two maps under which the system qubit state is transferred from the $+1$ into the -1 eigenspace of $Y_1 Y_2$ and $X_1 X_2$.

In order to implement the first map, three unitary operations and a dissipative one have been used. All act on the two system qubits, along with the ancillary control qubit, denoted as before with the subscript c . A key role is played by so-called Mølmer-Sørensen (MS) entangling gates [14], of the form $U_{X^2}(\theta) = \exp(-i\frac{\theta}{4}S_x^2)$ and $U_{Y^2}(\theta) = \exp(-i\frac{\theta}{4}S_y^2)$, where S_x and S_y denote collective spin operators, $S_x \equiv \sum_i X_i$ and $S_y \equiv \sum_i Y_i$. The mapping steps of the experimental pumping protocols are then as follows:

- (i) Information about whether the system is in the $+1$ or -1 eigenspace of $X_1 X_2$ is mapped by the MS gate $U_{X^2}(\pi/2)$ on $\mathcal{H}_c \otimes \mathcal{H}_Q$ onto the logical states $|0\rangle$ and $|1\rangle$ of the ancilla (initially in $|1\rangle$).
- (ii) A controlled gate performs a conversion from the $+1$ eigenvalue of the stabilizer $X_1 X_2$ to -1 by acting on the ancilla and the first system qubits³:

$$C_{\text{in}} = |0\rangle\langle 0|_c \otimes Z_1 + |1\rangle\langle 1|_c \otimes I.$$

- (iii) The MS gate $U_{X^2}(\pi/2)$ is re-applied, in order to move the state back to the initial basis representation.

- (iv) The ancilla qubit is dissipatively reset to state $|1\rangle$.

Next, the above cycle is repeated, this time using $U_{Y^2}(\pi/2)$ gates in steps (i) and (iii), with the controlled-gate C_{in} remaining unchanged.

³In the experiment, the following sequence is employed: $C_{\text{in}} = C(p) = U_{Z_1}(\alpha)U_Y(\pi/2)U_{X^2}^{(c,1)}(-\alpha)U_Y(-\pi/2)$, with $U_{X^2}^{(c,1)}(-\alpha) = \exp(i(\alpha/2)X_c X_1)$, $U_{Z_1}(\alpha) = e^{i\alpha Z_1}$, and $p = \sin^2(\alpha)$.

4.2 Bell-state pumping and splitting subspaces

In order to show that the above stabilization scheme can be seen as an instance of our splitting-subspace approach, it is necessary to take a closer look at the structure of the MS gate. The MS entangling gate is based on pairwise interaction terms and for the present discussion it suffices to consider the explicit form already given earlier⁴, that is,

$$U_{X^2}(\theta) = \exp\left(-i\frac{\theta}{4}S_x^2\right) \equiv \exp\left(-i\frac{\theta}{4}\left(\sum_i X_i\right)^2\right), \quad (8)$$

where the sum defining the collective spin operators S_x is understood to be performed over all the ions involved in the gate. In the experimental Bell-state pumping protocol we have just examined, MS gates with a phase angle $\theta = \pi/2$ are required. In this case, the $U_{X^2}(\pi/2)$ operator in (8) can be decomposed in a more explicit form with respect to the relevant $\mathcal{H}_c \otimes \mathcal{H}_Q$ tensor-product decomposition:

$$U_{X^2}\left(\frac{\pi}{2}\right) = U'_X(I \otimes \Pi_{-1} + X \otimes \Pi_{+1}),$$

where U'_X is a unitary 8×8 matrix of the form:

$$U'_X = I_2 \otimes \frac{1}{2}e^{-i\pi/8} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \equiv I_2 \otimes U''_X.$$

The action of the unitary operator U''_X on the Bell states basis (7) is described as:

$$\begin{aligned} U''_X|\Phi^+\rangle &= -\frac{1}{\sqrt{2}}e^{-i\pi/8}|\Psi^+\rangle, \\ U''_X|\Phi^-\rangle &= \frac{1}{\sqrt{2}}e^{-i\pi/8}|\Phi^-\rangle, \\ U''_X|\Psi^+\rangle &= -\frac{1}{\sqrt{2}}e^{-i\pi/8}|\Phi^+\rangle, \\ U''_X|\Psi^-\rangle &= \frac{1}{\sqrt{2}}e^{-i\pi/8}|\Psi^-\rangle. \end{aligned}$$

⁴In its most general form, the MS gate can be parametrized by two angles θ and ϕ , $U_{\text{MS}}(\theta, \phi) = \exp\left(-i\frac{\theta}{4}(\cos\phi S_x + \sin\phi S_y)^2\right)$.

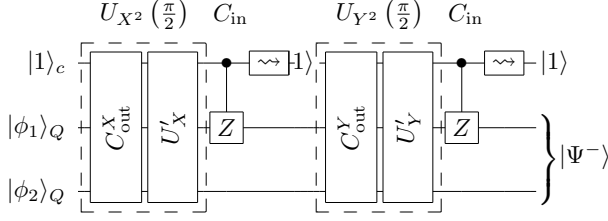


Figure 2: Simplified version (with a *single* MS gate being employed in each control cycle) of the Bell-state pumping protocol experimentally implemented in trapped ions by Barreiro *et al.* [3].

Hence, U''_X is *not* harmful for our purpose since it swaps the Bell states in the $+1$ eigenspace of X_1X_2 and does not change (except for an irrelevant constant), the other Bell states in the -1 eigenspace. Explicitly, in the Bell basis $\mathcal{B}_{\text{Bell}} := \{|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle\}$, U''_X can be written as:

$$U''_{X,\text{Bell}} = \frac{1}{\sqrt{2}} e^{-i\pi/8} \left[\begin{array}{c|c} -X & O_2 \\ \hline O_2 & I_2 \end{array} \right],$$

where the symbol O_2 denotes a 2×2 matrix of zeros. We have thus obtained a decomposition of the MS gate $U_{X^2}(\pi/2)$ in a form that includes two terms:

1. The conditional gate $C^X_{\text{out}} = I \otimes \Pi_{-1} + X \otimes \Pi_{+1}$, that coherently transfers to the ancilla qubit the information on which of the two subspaces the system's state is in;
2. An additional unitary U'_X that is not harmful for stabilization purposes, since it commutes with the projector onto the eigenspaces of X_1X_2 .

This means that the $U_{X^2}(\pi/2)$ gate can be decomposed precisely in the form for \tilde{C}_{out} given in Eq. (4), and is thus a viable operation for implementing a splitting-subspace approach to stabilize the desired subspace.

Similarly, the MS gate $U_{Y^2}(\pi/2)$ can be decomposed in the form:

$$U_{Y^2}\left(\frac{\pi}{2}\right) = U'_Y(Z \otimes \Pi'_{-1} + X \otimes \Pi'_{+1}),$$

where now the unitary operator U'_Y reads:

$$U'_Y = Z \otimes \frac{1}{2} e^{-i\pi/8} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \equiv Z \otimes U''_Y.$$

Thus, we have likewise found a decomposition of the MS gate $U_{Y^2}(\pi/2)$ in the form of Eq. (4), involving a conditional unitary $C^Y_{\text{out}} = Z \otimes \Pi'_{-1} + X \otimes \Pi'_{+1}$ and an additional unitary U'_Y with the same roles as for the X_1X_2 case. This means that also $U_{Y^2}(\pi/2)$ is an admissible entangling operation for the splitting-subspace method.

Note that in the implementation of the experimental protocol described in Sec. 4.1, the *second* application of the MS gates in step (iii) has essentially the effect of canceling the action of U'_X, U'_Y on the target qubits. However, it is easy to see that for the first map, there is no actual need for this second MS gate. In fact, the net effect of the “residual” U'_X is to swap the states that were in the $+1$ eigenspace of X_1X_2 , but they end up being pumped in the correct subspace anyway. Similarly, one can check by direct calculation that the action of U'_Y on the subspace that is prepared by the first map, namely the one generated by $\{|\Phi^-\rangle, |\Psi^-\rangle\}$, is:

$$\begin{aligned} U''_Y|\Phi^-\rangle &= -\frac{1}{\sqrt{2}} e^{-i\pi/8} |\Psi^+\rangle, \\ U''_Y|\Psi^-\rangle &= \frac{1}{\sqrt{2}} e^{-i\pi/8} |\Psi^-\rangle. \end{aligned}$$

The desired state $|\Psi^-\rangle$ is not perturbed: accordingly, if we now apply the controlled- σ_z to the other state we obtain:

$$Z_1|\Psi^+\rangle = |\Psi^-\rangle,$$

and the desired state preparation is achieved without the need for step (iii) or, equivalently, without undoing U'_X, U'_Y . This simplified version of the stabilization protocol, which in fact corresponds to the experimentally implemented one as noted in the Supplementary Material [3], is depicted in Fig. 2.

5 Conclusions

We presented a general framework to design stabilizing controls that prepare *arbitrary pure states on qubit registers in finite time*: once a representation of the target state in terms of splitting subspaces is chosen, the control objective can be achieved either by measurements and discrete-time *classical feedback* [4, 5], or by resorting to *coherent feedback* via quantum controllers [7]. In the latter case, the measurement and feedback steps are effectively replaced by suitable conditional operations ($C_{\text{out}}, C_{\text{in}}$ in our setting), which enable the required quantum-information flows out of the system to the controller, and viceversa. This fully coherent implementation can be practically advantageous in a number of situations where quantum measurements are exceedingly slow and/or inaccurate, or destructive for the system itself.

The proposed splitting-subspace approach leaves, in its current form, significant freedom in constructing a suitable basis for the system state space. While we have shown how the latter translates in added flexibility for implementing the required conditional operations and can be exploited in principle to minimize the complexity of the stabilization protocol, systematic techniques for *optimizing* the generation of a splitting-subspace decomposition in specific control settings remain an interesting problem for future studies. We have illustrated our general approach by revisiting a recently proposed protocol for dissipative Bell-state preparation in ion traps [3] in the light of an explicit splitting-subspace analysis, demonstrating its direct applicability to current experimental situations. We expect that this will pave the way to further applications of our ideas to synthesizing finite-time dissipative state-preparation protocols in other qubit devices and control scenarios of experimental relevance.

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